

# A note on multipliers of discrete quantum groups

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## Abstract

We investigate the problem whether a given multiplier of a tensor product of two algebras belongs to the tensor product of multiplier algebras. We give a characterization of such multipliers in the case when one of the algebras is the algebra of functions on a discrete quantum group.

## 1 Introduction

Let  $\mathcal{A}$  be an algebra (over  $\mathbb{C}$ ) with non degenerate product. Then we can define the multiplier algebra  $M(\mathcal{A})$  as the vector space of pairs  $(\lambda, \rho)$  of linear maps  $\mathcal{A} \rightarrow \mathcal{A}$  such that

$$\rho(a)b = a\lambda(b)$$

for all  $a, b \in \mathcal{A}$ . It is customary to treat  $\lambda$  and  $\rho$  as left and right multiplication by an auxiliary object  $m$ , i.e.  $\lambda(a) = ma$  and  $\rho(a) = am$  for all  $a \in \mathcal{A}$ . The element  $m$  is called a *multiplier* of  $\mathcal{A}$ , while the pair  $(\lambda, \rho)$  is traditionally referred to as a *double centralizer* of  $\mathcal{A}$ . Usually the a multiplier  $m$  and the map  $\lambda$  corresponding to left multiplication by  $m$  are denoted by the same symbol. Note that if  $\mathcal{A}$  has a unit then  $M(\mathcal{A}) = \mathcal{A}$ .

If  $\mathcal{B}$  is another algebra with non degenerate product then  $\mathcal{B} \otimes \mathcal{A}$  is again an algebra with non degenerate product ([4, Lemma A.2]) and we can define the multiplier algebra  $M(\mathcal{B} \otimes \mathcal{A})$ . The tensor product of multiplier algebras of  $\mathcal{B}$  and  $\mathcal{A}$  embeds naturally into  $M(\mathcal{B} \otimes \mathcal{A})$  ([4, Prop. A.3]). In [3] the image of this embedding was used to characterize almost periodic elements for a discrete quantum group and helped in constructing the analogue of Bohr compactification for discrete quantum groups. Our aim in the present paper is to provide a criterion characterizing elements of the image of this embedding.

For any vector space  $V$  we shall denote by  $V^\#$  the space of all linear functionals on  $V$ . If  $\mathcal{A}$  is an algebra then  $\mathcal{A}^\#$  is an  $\mathcal{A}$ -bimodule in a natural way: for  $f \in \mathcal{A}^\#$  and  $a \in \mathcal{A}$

$$\begin{aligned}(af)(b) &= f(ba), \\ (fa)(b) &= f(ab)\end{aligned}$$

for all  $b \in \mathcal{A}$ . An important subspace of  $\mathcal{A}^\#$  is the space of *reduced functionals* on  $\mathcal{A}$  which is by definition

$$\mathcal{A}^* = \text{span}\{afb : a, b \in \mathcal{A}, f \in \mathcal{A}^\#\}.$$

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Any functional in  $\mathcal{A}^*$  admits a natural extension to  $M(\mathcal{A})$ . If  $\mathcal{B}$  is another algebra with non degenerate product then for all  $\zeta \in \mathcal{B}^*$  and  $\xi \in \mathcal{A}^*$  the tensor product  $\zeta \otimes \xi$  is a reduced functional on  $\mathcal{B} \otimes \mathcal{A}$  and so it extends to  $M(\mathcal{B} \otimes \mathcal{A})$ . We have the following simple result ([3]):

**Proposition 1.1** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be algebras with non degenerate products and let  $Y$  be a multiplier of  $\mathcal{B} \otimes \mathcal{A}$ . Then for any  $\xi \in \mathcal{A}^*$  there exists a unique multiplier  $m \in M(\mathcal{B})$  such that*

$$(\zeta \otimes \xi)(Y) = \zeta(m)$$

for all  $\zeta \in \mathcal{B}^*$ .

The multiplier  $m$  constructed in Proposition 1.1 is called a *right slice of  $Y$  with  $\xi$*  and will be denoted by  $(\text{id} \otimes \xi)(Y)$ .

Multiplier algebras have been introduced into the theory of Hopf algebras by Van Daele in [4] where the notion of a multiplier Hopf algebra was defined. The theory of multiplier Hopf algebras provides a natural framework to study a variety of quantum groups. In particular discrete quantum groups have been studied in this framework in [5].

A multiplier Hopf algebra is a pair  $(\mathcal{A}, \delta)$  of an algebra with non degenerate product and a homomorphism  $\delta: \mathcal{A} \rightarrow M(\mathcal{A} \otimes \mathcal{A})$  such that the maps

$$\begin{aligned} T_1 : \mathcal{A} \otimes \mathcal{A} \ni (a \otimes b) &\longmapsto \delta(a)(I \otimes b), \\ T_2 : \mathcal{A} \otimes \mathcal{A} \ni (a \otimes b) &\longmapsto (I \otimes a)\delta(b) \end{aligned}$$

have image equal to  $\mathcal{A} \otimes \mathcal{A}$ , are bijective onto this image and  $T_2 \otimes \text{id}$  and  $\text{id} \otimes T_1$  on  $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$  commute. By composing  $\delta$  with an extension of the flip map  $(a \otimes b \mapsto b \otimes a)$  to  $M(\mathcal{A} \otimes \mathcal{A})$  we obtain another homomorphism  $\delta'$ . If  $(\mathcal{A}, \delta')$  is a multiplier Hopf algebra then the multiplier Hopf algebra  $(\mathcal{A}, \delta)$  is called *regular*. If  $\mathcal{A}$  is a  $*$ -algebra and  $\delta$  is a  $*$ -homomorphism (in this case  $M(\mathcal{A})$  carries a natural involution) then we say that  $(\mathcal{A}, \delta)$  is a multiplier Hopf  $*$ -algebra.

A *discrete quantum group* is a multiplier Hopf  $*$ -algebra  $(\mathcal{A}, \delta)$  such that  $\mathcal{A}$  is a direct sum of a family of full matrix algebras. Since multiplier Hopf  $*$ -algebras are automatically regular ([4, Sect. 5]) we see that discrete quantum groups are regular multiplier Hopf algebras. We shall not make use of the involutive structure of discrete quantum groups, but regularity will be of importance for our results.

Discrete quantum groups appeared first in the C\*-algebraic context in [2].

Let  $(\mathcal{A}, \delta)$  be a multiplier Hopf algebra. A functional  $\varphi$  on  $\mathcal{A}$  is called *left invariant* if the map  $\mathcal{A} \ni a \mapsto (\text{id} \otimes \varphi)\delta(a) \in M(\mathcal{A})$  satisfies

$$(\text{id} \otimes \varphi)\delta(a) = \varphi(a)I.$$

If a non trivial left invariant functional exists then it is unique up to multiplication by a constant. Moreover if a left invariant functional exists then so does a right invariant functional (defined similarly with an obvious modification). Discrete quantum groups have invariant functionals ([5]) and we will freely use the results of the theory of regular multiplier Hopf algebras with invariant functionals developed in [6].

## 2 Reduced functionals as multipliers

Let  $(\mathcal{A}, \delta)$  be a multiplier Hopf algebra. It was proved in [4, Prop. 6.2] that the comultiplication  $\delta$  defines an associative algebra structure on  $\mathcal{A}^*$  by

$$(\xi_1 \star \xi_2)(a) = (\xi_1 \otimes \xi_2)(\delta(a))$$

for all  $\xi_1, \xi_2 \in \mathcal{A}^*$  and all  $a \in \mathcal{A}$ . We call this multiplication the *convolution product* of reduced functionals.

Assume now that  $(\mathcal{A}, \delta)$  is a regular multiplier Hopf algebra with (non trivial) invariant functionals and let  $\varphi$  be a left invariant functional on  $\mathcal{A}$ . It is known ([6, Prop. 4.2]) that the set

$$\hat{\mathcal{A}} = \{a\varphi : a \in \mathcal{A}\} = \{\varphi a : a \in \mathcal{A}\}$$

is also an associative algebra under the convolution product and this product is non degenerate.

**Proposition 2.1** *Let  $(\mathcal{A}, \delta)$  be a regular multiplier Hopf algebra with invariant functionals. Then  $\hat{\mathcal{A}}$  embeds into  $\mathcal{A}^*$  and  $\mathcal{A}^*$  is a subalgebra of the multiplier algebra of  $\hat{\mathcal{A}}$ .*

PROOF. For the first statement we use the fact that  $\mathcal{A} = \mathcal{A}^2$  (this is true for any multiplier Hopf algebra) and the existence of the modular automorphism for  $\varphi$ , i.e. an automorphism  $\sigma$  of  $\mathcal{A}$  such that

$$\varphi a = \sigma(a)\varphi$$

for all  $a \in \mathcal{A}$  ([6, Prop. 3.12]). Let  $b \in \mathcal{A}$  and write  $b$  as

$$b = \sum_{k=1}^N r_k s_k$$

with  $r_k, s_k \in \mathcal{A}$ . Then

$$\varphi b = \sum_{k=1}^N \sigma(r_k) \varphi s_k \in \mathcal{A}^*.$$

Now that we have established that  $\hat{\mathcal{A}} \subset \mathcal{A}^*$  we shall use a standard technique to show that  $\hat{\mathcal{A}}$  is in fact an ideal in  $\mathcal{A}^*$ . Let  $a, b, c \in \mathcal{A}$  and  $f \in \mathcal{A}^\sharp$ . Set  $\gamma = a f b$ ,  $\xi = c \varphi$ . Further write

$$a \otimes c = \sum_{k=1}^N \delta(r_k)(s_k \otimes I)$$

with  $r_k, s_k \in \mathcal{A}$  for  $k = 1, \dots, N$  (this is possible for a regular multiplier Hopf algebra). Then we have

$$\begin{aligned} (\gamma \star \xi)(x) &= (a f b \otimes c \varphi) \delta(x) \\ &= (f b \otimes \varphi)(\delta(x)(a \otimes c)) \\ &= (f b) \left[ (\text{id} \otimes \varphi) \sum_{k=1}^N \delta(x r_k)(s_k \otimes I) \right] \\ &= (f b) \left[ \sum_{k=1}^N [(\text{id} \otimes \varphi) \delta(x r_k)] s_k \right] \\ &= (f b) \left[ \sum_{k=1}^N \varphi(x r_k) s_k \right] \\ &= \sum_{k=1}^N f(b s_k) (r_k \varphi)(x) \end{aligned}$$

for any  $x \in \mathcal{A}$ , where in the second last equality we used the left invariance of  $\varphi$ . This way we showed that  $\gamma \star \xi$  belongs to  $\hat{\mathcal{A}}$ . For the product  $\xi \star \gamma$  we use the fact that any element of  $\hat{\mathcal{A}}$  can be written as  $d\psi$  with  $\psi$  a right invariant functional. Then the argument can be repeated with the difference that  $d \otimes a$  has to be written as

$$d \otimes a = \sum_{k=1}^N \delta(p_k)(I \otimes q_k)$$

with  $p_k, q_k \in \mathcal{A}$ ,  $k = 1, \dots, N$ .

Now the associativity of the (convolution) product in  $\mathcal{A}^*$  shows that

$$\xi_1 \star (\gamma \star \xi_2) = (\xi_1 \star \gamma) \star \xi_2$$

for  $\gamma \in \mathcal{A}^*$  and  $\xi_1, \xi_2 \in \widehat{\mathcal{A}}$ . This means that  $\gamma$  is a multiplier of  $\widehat{\mathcal{A}}$ .

Q.E.D.

### 3 A characterization of tensor products of multipliers

Let  $(\mathcal{A}, \delta)$  be a discrete quantum group. Then the algebra  $\widehat{\mathcal{A}}$  has a unit ([6, Prop. 5.3]) and consequently  $\widehat{\mathcal{A}} = \mathcal{A}^*$ . If in addition  $(\mathcal{A}, \delta)$  is a discrete quantum group, it is easy to see that

$$\mathcal{A}^* = \text{span} \{af : f \in \mathcal{A}^\sharp, a \in \mathcal{A}\} = \text{span} \{fa : f \in \mathcal{A}^\sharp, a \in \mathcal{A}\}.$$

In particular for a discrete quantum group  $(\mathcal{A}, \delta)$  we have

$$\widehat{\mathcal{A}} = \text{span} \{af : f \in \mathcal{A}^\sharp, a \in \mathcal{A}\} = \text{span} \{fa : f \in \mathcal{A}^\sharp, a \in \mathcal{A}\}. \quad (1)$$

Indeed for  $a, b \in \mathcal{A}$  and  $f \in \mathcal{A}^\sharp$  we have  $af = afe_a$  and  $fb = e_bfb$  where  $e_a$  and  $e_b$  are, for example, the units of the ideals generated by  $a$  and  $b$  respectively. Property (1) is crucial for our next result.

**Theorem 3.1** *Let  $(\mathcal{A}, \delta)$  be a discrete quantum group and let  $\mathcal{B}$  be an algebra with non degenerate product. Let  $Y$  be a multiplier of  $\mathcal{B} \otimes \mathcal{A}$ . Then*

$$\left( Y \in M(\mathcal{B}) \otimes M(\mathcal{A}) \right) \iff \left( \dim \left\{ (\text{id} \otimes \xi)(Y) : \xi \in \widehat{\mathcal{A}} \right\} < \infty \right) \quad (2)$$

PROOF. The implication “ $\Rightarrow$ ” is straightforward. Assume that the right hand side of (2) holds. Then let  $\{x_1, \dots, x_N\}$  be a basis in the space of right slices of  $Y$ . Fix  $b \in \mathcal{B}$  and  $a \in \mathcal{A}$ . We shall consider the expression

$$(\text{id} \otimes \varphi)(Y(b \otimes a)) = ((\text{id} \otimes a\varphi)Y)b. \quad (3)$$

There exist unique scalars  $\lambda_k(a)$  such that the right hand side of (3) is equal to

$$\left( \sum_{k=1}^N \lambda_k(a) x_k \right) b = \sum_{k=1}^N \lambda_k(a) x_k b. \quad (4)$$

Clearly each  $\lambda_k$  is a linear functional on  $\mathcal{A}$ . By (1) for any  $c \in \mathcal{A}$  we have  $\lambda_k c \in \widehat{\mathcal{A}}$ . By [6, Lemma 4.11] and the biduality theorem ([6, Thm. 4.12]) that the map

$$\widehat{\mathcal{A}} \ni d\varphi \longmapsto (\lambda_k c)(d) = \lambda_k(cd)$$

is determined by a unique element  $q \in \mathcal{A}$  in such a way that

$$\lambda_k(cd) = (d\varphi)(q). \quad (5)$$

Writing  $q = y_k(c)$  we define linear maps  $y_k : \mathcal{A} \rightarrow \mathcal{A}$  for  $k = 1, \dots, N$ .

Consider now

$$(\text{id} \otimes \varphi)((b \otimes a)Y) = b((\text{id} \otimes \varphi a)Y) = b((\text{id} \otimes \sigma(a)\varphi)Y).$$

As before we write this last expression as

$$b \left( \sum_{k=1}^N \lambda_k(\sigma(a)) x_k \right) = \sum_{k=1}^N \lambda_k(\sigma(a)) b x_k$$

and we can define linear maps  $z_k: \mathcal{A} \rightarrow \mathcal{A}$  by

$$(\varphi d)(z_k(c)) = \lambda_k(\sigma(dc))$$

or in other words

$$(d\varphi)(z_k(c)) = \lambda_k(d\sigma(c)).$$

We shall show that  $((y_k, z_k))_{k=1, \dots, N}$  are double centralizers of  $\mathcal{A}$ . Take  $a_1, a_2 \in \mathcal{A}$  and  $d\varphi \in \widehat{\mathcal{A}}$ . We have

$$\begin{aligned} (d\varphi)(z_k(a_1)a_2) &= ((a_2d)\varphi)(z_k(a_1)) = \lambda_k(a_2d\sigma(a_1)) \\ (d\varphi)(a_1y_k(a_2)) &= \varphi(a_1y_k(a_2)d) = \varphi(y_k(a_2)d\sigma(a_1)) \\ &= [(d\sigma(a_1))\varphi](y_k(a_2)) = \lambda_k(a_2d\sigma(a_1)) \end{aligned}$$

and since this equality holds for any  $d\varphi \in \widehat{\mathcal{A}}$  we have that  $z_k(a_1)a_2 = a_1y_k(a_2)$  for all  $a_1, a_2 \in \mathcal{A}$ . This way we have obtained multipliers  $(y_k)_{k=1, \dots, N}$  of  $\mathcal{A}$ .

Upon substitution of  $y_k c$  for  $q$  in equation (5) we obtain

$$\lambda_k(cd) = \varphi(y_k cd)$$

for all  $c, d \in \mathcal{A}$ . Inserting  $c = a$  and  $d$  equal to the unit of the ideal generated by  $a$  we obtain  $\lambda_k(a) = \varphi(y_k a)$  for  $k = 1, \dots, N$ . Therefore (cf. (3) and (4)) the multipliers  $(y_k)_{k=1, \dots, N}$  satisfy

$$(\text{id} \otimes \varphi) \left[ \left( \sum_{k=1}^N x_k \otimes y_k \right) (b \otimes a) \right] = (\text{id} \otimes \varphi)[Y(b \otimes a)].$$

As  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$  were arbitrary we obtain this equality for all  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ . By faithfulness of  $\varphi$  (cf. [6, Sect. 3]) we have

$$Y = \sum_{k=1}^N x_k \otimes y_k$$

and consequently  $Y \in M(\mathcal{B}) \otimes M(\mathcal{A})$ .

Q.E.D.

In [3] the following definition of an almost periodic element for a discrete quantum group  $(\mathcal{A}, \delta)$  was proposed: a multiplier  $x \in M(\mathcal{A})$  is an almost periodic element for  $(\mathcal{A}, \delta)$  if  $\delta(x) \in M(\mathcal{A}) \otimes M(\mathcal{A})$  (where we use the unique extension of  $\delta$  to the multiplier algebra). It was shown that the set of all almost periodic elements for  $(\mathcal{A}, \delta)$  forms a Hopf  $*$ -algebra. On the formal level the definition of an almost periodic element is similar to that of an almost periodic function on a locally compact group (cf. [1, §41]). Theorem 3.1 shows that this definition corresponds more directly to the notion of an *almost invariant* function ([1, Sect. 39D]), but this distinction fades on the purely algebraic level.

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